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ON A.E. CONVERGENCE OF SOLUTIONS OF HYPERBOLIC EQUATIONS TO L^p-INITIAL DATA

BY

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ABSTRACT. We consider the Cauchy data problem $u(x,0) = 0$, $\partial u(x,0)/\partial t = f(x)$, for a strongly hyperbolic second order equation in nth spatial dimension, $n \geq 3$, with C^∞ coefficients. Almost everywhere convergence of the solution of this problem to initial data, in the appropriate sense is proved for f in L^p, $2n/(n+1) < p < 2(n-2)/(n-3)$. The basic techniques are L^p-estimates for some maximal operators associated to the problem (see [**4**]), and the asymptotic expansion of the Riemann function given by D. Ludwig (see [**9**]).

Introduction. In [**11**] Stein and Wainger study the behavior of the maximal function given by averages over dilates of the unit sphere ($n \geq 3$) by using harmonic analysis methods; the a.e. convergence of the wave equations to L^p-initial data, for some range of p's, is a consequence of their maximal theorem: this method is generalized by Greenleaf [**4**] to "variable coefficient" spheres ($n = 3$), taking the parameter of dilation in a neighborhood of the origin, and, as a consequence, he obtains a.e. convergence to L^p-initial data for $p > 3/2$ for the "wave operator" given by the Laplace-Beltrami operator on a 4-dimensional manifold with real analytic Lorentzian metric. He uses a formula by M. Riesz [**10**], valid in this particular case.

In the present paper we obtain a.e. convergence to L^p-initial data ($2n/(n+1) < p < 2(n-2)/(n-3)$) for the solution of a strongly hyperbolic n-dimensional equation with C^∞ coefficients by using the Stein-Wainger-Greenleaf method and an asymptotic expansion (generalized Huygen's principle) for the solution given in [**9**].

For the sake of completeness we give (Appendix 1) the definition of the analytic extension we are going to use, following [**3**], and in Appendix 2 we sketch the proof of the asymptotic expansion in [**9**], based on some of the steps in [**8** and **9**], in order to point out some properties we need for our purposes. The reader may avoid both appendices.

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Let us consider second order equations in $n+1$ variables $\overline{x} = (x_0, x_1, \ldots, x_n)$,

$$Lu = \sum_{i,j=0}^{n} a_{ij}(\overline{x}) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{k=0}^{n} b_k(\overline{x}) \frac{\partial u}{\partial x_k} + c(\overline{x}) u, \tag{0}$$

where the a_{ij}'s, b_R's and c are real C^∞ functions defined in an open set $D \subset \mathbf{R}^{n+1}$.

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DEFINITION 1. L is strongly hyperbolic with respect to the direction $\xi \neq 0$ at a point y iff, for any arbitrary θ nonparallel to ξ, the equation in λ,

$$Q(\lambda\xi + \theta) = \sum a_{ij}(y)(\lambda\xi_i + \theta_i)(\lambda\xi_j + \theta_j) = 0,$$

has two distinct real roots. Such a ξ is said to be a timelike direction at y. L is strongly hyperbolic at a point y when there exists a timelike direction at y.

From now on we consider L strongly hyperbolic at every point $y \in D$.

DEFINITION 2. A C^∞ hypersurface S of $\mathbf{R}^{n+1}$ is said to be spacelike if the normal η_x to S at any point $x \in S$ is a timelike direction at x.

The problem to be treated here is the initial value

$$(1') \qquad Lu = 0, \quad u|_S = 0, \quad \partial u/\partial\eta_x = f(x),$$

where S is spacelike, $f \in C^\infty(S)$, and D is an open domain containing S.

LEMMA 3. *If S is a spacelike surface, one can construct, in a neighborhood of S, a smooth real-valued function $t(\overline{x})$ such that S is a level surface for t and every level surface of t is spacelike for L. Furthermore, L, expressed in terms of the new coordinates $(t, x_1^1, \ldots, x_n^2)$, is strongly hyperbolic and the surfaces t =constant are spacelike. By an abuse of language we keep the notation $(t, x_1, \ldots, x_n) = (t, x)$.*

REMARK. Lemma 3 allows us to reduce the initial value problem (1′) to (1), where $\overline{f}$, $\overline{u}$ and $\overline{L}$ are expressed in the coordinates $(t, x_1, \ldots, x_n)$:

$$(1) \qquad \overline{L}\overline{u} = 0, \quad \overline{u}(0, x) = 0, \quad \partial\overline{u}(0, x)/\partial t = \overline{f}(x).$$

In what follows, t denotes a variable such that the surface $t =$ constant is spacelike for $0 \leq t < t_0$.

DEFINITION 4. The Riemann function $R(t, x; z)$ is the solution of the initial value problem for L as in (1):

$$(2) \qquad Lu = 0, \quad u(0, x) = 0, \quad \partial u(0, x)/\partial t = \delta(x - z).$$

Leray [**7**] proved the existence and uniqueness of $R(t, x; z)$ in the C^∞ coefficients case, Hadamard [**5**] in the analytic case; also Ludwig [**8**] proved Hadamard's result through the convergence of the asymptotic wave expansion for $R(t, x; z)$.

THEOREM 5 (LUDWIG: PROGRESSIVE WAVE SOLUTION FOR (2)).

$$R(t, x; z) = \sum_{j=0}^{N} t^{-(n+1)/2} \frac{(\phi(t, x, z))_+^{-(n-2)/2+j}}{\Gamma(-(n-3)/2 + j)} \cdot \tilde{a}_j(t, x; z) + \tilde{R}_N(t, x, z),$$

where $\tilde{R}_N$ can be made as smooth as desired by taking N large enough, it is defined for every $t \in [0, t_0]$, and it is supported as a function of x in a ball of $\mathbf{R}^n$ of radius $O(t)$ for $t \to 0$ (fixed z, $O(t)$ may depend on z). The same statement holds for the z-variable when x is fixed.

$\phi(t, x; z)$ can be written as

$$t - \frac{|x - z|}{\tilde{r}(t, x; (x - z)/|x - z|)},$$

where $\tilde{r}(t, x, \overline{y})$ is nonzero everywhere, defined for $x \in \mathbf{R}^n$ and $\overline{y} \in \Omega_{n-1}$ (the unit sphere in $\mathbf{R}^n$) and is C^∞; furthermore, the hypersurface $\{z\colon \phi(t, x; z) = 0\}$ has nonzero Gaussian curvature everywhere.

$\tilde{a}_j(t,x;z)$ are C^∞ functions with respect to $x, z \in \mathbf{R}^n$ and $t \in [0, t_0]$.

$\phi_+^\alpha/\Gamma(\alpha+1)$ is the α-analytic family of one real variable distributions defined in [**3**, *p.* 56].

COROLLARY 6. *The solution u of problem* (1) *is given by*

$$u(t,x) = \sum_{j=0}^{N} t^{-j} \left\langle \frac{\left(t^2 - \frac{|x-z|^2}{r^2(t,x,(x-z)/|x-z|)}\right)_+^{-(n-1)/2+j}}{\Gamma(-(n-3)/2+j)},\ f(z)a_j(t,x,z) \right\rangle_z + \frac{1}{t^{n-1}} \int_{\mathbf{R}^n} R_N(t,x,z) f(z)\, dz,$$

where $\langle\ ,\ \rangle_z$ denotes z-distribution applied to a function, and r, a_j and R_N satisfy the same assumption as $\tilde{r}, \tilde{a}_j$ and $\tilde{R}_N$ in Theorem 5 (*see Appendix* 2).

DEFINITION 7. For $f \in S(\mathbf{R}^n)$ and $t \in (0, t_0]$, $0 \le u \le 1$,

$$\left[M_{t,u,j}^\alpha f(x) = \left\langle \frac{(tu)^{-2\alpha-(n-2)}}{\Gamma(\alpha)} \left((tu)^2 - \frac{|y|^2}{r^2(t,x,y/|y|)}\right)_+^{\alpha-1}, f(x-y)a_j(t,x,x-y) \right\rangle_y \right].$$

This is an α-analytic family of operators.

DEFINITION 7′. Let α_0 be fixed, $\alpha_0, \alpha \in \mathbf{C}$, $\operatorname{Re}\alpha_0 > (1-n)/2$,

$$T_{t,j}^\alpha = \frac{\text{reg}}{\Gamma(\alpha-\alpha_0)} \int_0^1 M_{tv,1,j}^{\alpha_0} f(x) v^{n-1+2\alpha_0}(1-v^2)^{\alpha-\alpha_0-1}\, dv.$$

(See Appendix 1 for the relevant definitions.)

LEMMA 8. *For* $\operatorname{Re}\alpha' < \operatorname{Re}\alpha$, $\operatorname{Re}\alpha' > -(n+1)/2$,

$$M_{t,1,j}^\alpha f(x) = \frac{1}{\Gamma(\alpha-\alpha')} \int_0^1 M_{t,u,j}^{\alpha'} f(x) u^{n-1+2\alpha'}(1-u^2)^{\alpha-\alpha'-1}\, du.$$

PROOF. We are going to prove the case $\operatorname{Re}\alpha' > 1$; the general case follows by analytic continuation.

$$\left\langle \frac{t^{-2\alpha-n+2}}{\Gamma(\alpha)} \left(t^2 - \frac{|y|^2}{r^2(t,x,y/|y|)}\right)_+^{\alpha-1}, f(x-y)a_j(t,x,x-y) \right\rangle_y$$
$$= \left\langle \frac{t^{-2\alpha-n+2}}{\Gamma(\alpha)} \left(t^2 - \frac{s^2}{r^2(t,x,\overline{y})}\right)_+^{\alpha-1}, f(x-s\overline{y})a_j(t,x,x-s\overline{y})s^{n-1} \right\rangle_{s,\overline{y}}.$$

After changing to polar coordinates $s = |y|$, $\overline{y} \in \Omega_{n-1}$. Let us take the Fourier transform with respect to the radius s (i.e. $\wedge_s$) of the even extension of the above functions: Since

$$\left(\frac{1}{\Gamma(\alpha)}\left(t^2 - \frac{s^2}{r^2(t,x,\overline{y})}\right)_+^{\alpha-1}\right)^{\wedge_s} = t^{2(\alpha-1)} \left[\frac{1}{\Gamma(\alpha)}\left(1 - \frac{s}{tr(t,x,\overline{y})}^2\right)_+^{\alpha-1}\right]^{\wedge_s}$$

is a function of $\overline{y} \in \Omega_{n-1}$ and $\sigma \in \mathbf{R}$ given by

$$\sqrt{\pi} t^{2(\alpha-1)} tr(t,x,\overline{y}) J_{\alpha-1/2}(\sigma tr(t,x,\overline{y})) \left(\frac{tr(t,x,\overline{y})\sigma}{\sigma}\right)^{-\alpha+1/2},$$

where the J's denote Bessel functions,

$$M^{\alpha}_{t,1,j} f(x) = t^{-n+1} \int_{y\in\Omega_{n-1}} r(t,x,\overline{y}) \int_{\sigma\in\mathbf{R}} \sqrt{\pi}\left(\frac{tr\sigma}{2}\right)^{1/2-\alpha} J_{\alpha-1/2}(\sigma tr) \cdot (f(x-s\overline{y}) a_j(t,x,x-sy) s^{n-1})^{\wedge_s}(\sigma)\, ds\sigma\, d\overline{y}.$$

By the recurrence integral formula for bessel functions, when $\operatorname{Re}\alpha' < \operatorname{Re}\alpha$ this equals

$$\begin{aligned}
&t^{-n+1}\int_{\overline{y}\in\Omega_{n-1}} r(t,x,\overline{y}) \int_{\sigma\in\mathbf{R}} \sqrt{\pi}\left(\frac{tr\sigma}{2}\right)^{1/2-\alpha} \frac{(rt\sigma)^{\alpha-\alpha'}}{2\Gamma(\alpha-\alpha')} \\
&\quad\cdot \int_0^1 J_{\alpha'-1/2}(\sigma tru) u^{\alpha'+1/2}(1-u^2)^{\alpha-\alpha'-1}\, du (f(x-s\overline{y}) a_j s^{n-1})^{\wedge_s}(\sigma)\, d\sigma\, d\overline{y} \\
&= \frac{t^{-n+1}}{\Gamma(\alpha-\alpha')} \int_0^1 \int_{\overline{y}\in\Omega_{n-1}} r(t,x,\overline{y}) \int_{\sigma\in\mathbf{R}} \sqrt{\pi}\left(\frac{t\sigma ru}{2}\right)^{1/2-\alpha'} J_{\alpha'-1/2}(\sigma tru) u^{2\alpha'} \\
&\quad\cdot (1-u^2)^{\alpha-\alpha'-1} (f(x-s\overline{y}) a_j s^{n-1})^{\wedge}(\sigma)\, d\sigma\, d\overline{y}\, du \\
&= \frac{t^{-n+1}}{\Gamma(\alpha-\alpha')} \int_0^1 \int_{\Omega_{n-1}} r(t,x,\overline{y}) \int_{\mathbf{R}} \left(\frac{1}{\Gamma(\alpha')}(1-s^2)_+^{\alpha'-1}\right)^{\wedge_s} (rtu\sigma) \\
&\quad\cdot (f(x-s\overline{y}) a_j s^{n-1})^{\wedge_s}(\sigma)\, d\sigma\, dy\, u^{2\alpha'}(1-u^2)^{\alpha-\alpha'-1}\, du \\
&= t^{-n}\int_0^1 \left\langle \frac{1}{\Gamma(\alpha')}\left(1-\frac{s^2}{(tur)^2}\right)_+^{\alpha'-1}, f(x-s\overline{y}) a_j(t,x,x-s\overline{y}) s^{n-1} \right\rangle_y \\
&\quad\cdot u^{2\alpha'+1}(1-u^2)^{\alpha-\alpha'-1}\, du \\
&= \frac{1}{\Gamma(\alpha-\alpha')}\int_0^1 \left\langle \frac{(tu)^{-n+2-2\alpha'}}{\Gamma(\alpha')}\left((ut)^2 - \frac{|y|^2}{r^2(t,x,y/|y|)}\right)_+^{\alpha'-1}, \right. \\
&\qquad \left. f(x-y) a_j(t,x,x-y) \right\rangle_y \\
&\quad\cdot u^{2\alpha'-1+n}(1-u^2)^{\alpha-\alpha'-1}\, du,
\end{aligned}$$

which is the statement of the lemma.

LEMMA 8′. $\operatorname{Re}\alpha' < \operatorname{Re}\alpha$,

$$T^{\alpha}_{t,j} f(x) = \frac{1}{\Gamma(\alpha-\alpha')}\int_0^1 T^{\alpha'}_{tu,j} f(x) u^{n+1+2\alpha'}(1-u^2)^{\alpha-\alpha'-1}\, du.$$

PROOF. It is a consequence of the formula $(\alpha,\beta>0)$

$$\int_a^b (a^2-s^2)^{\alpha-1}(b^2-s^2)^{\beta-1} s\, ds = \frac{1}{2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b^2-a^2)_+^{\alpha+\beta-1}$$

and analytic continuation to the whole range of α's.

REMARK 9. By Corollary 6 we can write

$$(1.1)\qquad u(x,t)=\sum_{j=0}^{N} t^{j+1} M_{t,1,j}^{-(n-2)/2+j} f(x)+\frac{1}{t^{n-1}}\int_{\mathbf{R}^n} R_N(t,x,z) f(z)\,dz.$$

Also notice that $T_{t,j}^{\alpha_0} f(x)=\frac{1}{2} M_{t,1,j}^{\alpha_0} f(x)$, since $M_{t,1,j}^{\alpha_0} f(x)$ is a smooth function of t (see Appendix 1, (A.3)).

L^p-boundedness and a.e. convergence. Let us consider the maximal operator

$$M_j^{\alpha_0} f(x)=\sup_{0<t\leq t_0}\left|M_{t,1,j}^{\alpha_0} f(x)\right|.$$

THEOREM 10. *For every compact set $K\subset\mathbf{R}^n$, there exists t_0 such that*

$$\left\|M_j^{\alpha_0} f\right\|_{L^p(K)}\leq C_{j,\alpha,K}\|f\|_p \quad \textit{for any } f$$

in the Schwartz class and $n/(\alpha+n-1)<p<-(n-2)/\alpha$ when $-(n-2)/2<\alpha<0$.

In order to prove this theorem, according to Greenleaf's version [**4**] of the methods for variable coefficients due to Stein and Wainger, we need Lemmas 11–15. Let us drop the index j to simplify notation.

LEMMA 11. *There exists a $t_0>0$ such that T_t^{α} is a Fourier integral operator for any $t<t_0$ and $\alpha>(1-n)/2$. Furthermore, its symbol is a function $b_{t,\alpha_0}^{\alpha}(x,t\xi)$ (smooth in t) in the class $S_{1,0}^{(1-n)/2-\alpha}(K_x\times\mathbf{R}_\xi^n)$, and its phase can be written as*

$$\phi_t(x,y,\xi)=(x-y)\xi+td(x,t,\xi)|\xi|,$$

where d is smooth for $x\in K$ and $t\in[0,t_0]$ and homogeneous of degree zero in ξ.

PROOF. Let us prove the statement in the case $\alpha=\alpha_0$, i.e. for $M_{t,1}^{\alpha_0}$. Our attempt is to reduce the expression for $M_{t,1}^{\alpha_0}$ to the explicit formula for $\delta^{(k)}$, for a suitable integer k, where $\delta^{(k)}$ is the kth derivative of Dirac's δ function.

For $\operatorname{Re}\alpha_0<0$ set $m=|\operatorname{Re}\alpha_0|$. Recall $m\geq -n/2$ when n is even and $m\geq -(n-1)/2$ when n is odd.

By Lemma 8,

$$\begin{aligned}M_{t,1}^{\alpha_0} f(x)&=c(\alpha_0)\int_0^1 M_{t,u}^{m} f(x)u^{n-1+2m}(1-u^2)^{\alpha_0-m-1}\,du\\ &=c\int_0^1\left\langle (ut)^{-2m-n+2}\delta^{(-m)}\left((ut)^2-\frac{|y|^2}{r^2(t,x,y/|y|)}\right), f(x-y)a(t,x,x-y)\right\rangle_y\\ &\qquad\qquad\cdot u^{n-1+2m}(1-u^2)^{\alpha_0-m-1}\,du.\end{aligned}$$

We may ignore the function a without loss of generality, for it is a C^∞ function. Hence we can write the above expression as

$$c(\alpha_0,m)\int_0^1\int_{\overline{y}\in\Omega_{n-1}} r^{2m}(t,x,\overline{y})\left(\frac{\partial}{\partial(s^2)}\right)^{-m}(f(x-s\overline{y})s^{n-2})\Big|\,d\sigma(\overline{y})_{s=utr}$$
$$\cdot (tu)^{-2m-n+2}u^{n-1+2m}(1-u^2)^{\alpha_0-m-1}\,du,$$

where

$$
\begin{aligned}
s &= |y| \\
&= \sum_{q=0}^{-m} c(\alpha_0, m) \int_0^1 \int_{\overline{y}\in\Omega_{n-1}} r^{-2m}(t,x,\overline{y}) \left(s^{n+2m-2+q} \frac{\partial^q f(x-sy)}{\partial s^q} \right) \Big| \, d\sigma(\overline{y})_{s=utr} \\
&\qquad \cdot (tu)^{-2m-n+2} u^{n-1+2m} (1-u^2)^{\alpha_0 - m - 1} \, du \\
&= \sum_{q=0}^{-m} c(\alpha_0, m) \int_0^1 \int_{\overline{y}\in\Omega_{n-1}} r^{n-2+q} \sum_{|\beta|=0}^{q} p_\beta(\overline{y}) \partial^\beta f(x - s\overline{y}) \Big|_{s=|y|=utr} \, d\sigma(\overline{y}) \\
&\qquad \cdot (tu)^q u^{-1+2m+n} (1-u^2)^{\alpha_0 - m - 1} \, du,
\end{aligned}
$$

where β is a multi-index $(\beta_1, \beta_2, \ldots, \beta_n)$, and $\partial^\beta = \partial^{|\beta|}/\partial y_1^{\beta_1} \partial y_2^{\beta_2} \cdots \partial y_n^{\beta_n}$, p_β is homogeneous of degree 0. Then

$$
\begin{aligned}
(1.2) \quad &= \sum_{q=0}^{-m} c(\alpha_0, m) \int_0^1 \int_{\Omega_{n-1}} \sum_{|\beta|\le q} r^{n-2+|q|} t^{|q|} p_\beta(\overline{y}) \partial^\beta \,|_{|y|=tur} f(x-y) \, d\sigma(\overline{y}) \\
&\qquad \cdot u^{|q|+2m+n-1} (1-u^2)^{\alpha+0-m+1} \, du \\
&= \sum_{|\beta|\le -m} c(\alpha_0, m) \int_0^1 \left\langle d\theta^\beta_{x,t,u}(y), t^{|\beta|} \partial^\beta f(x \cdot y) \right\rangle_y \\
&\qquad \cdot u^{|\beta|+2m+n-1} (1-u^2)^{\alpha_0 - m + 1} \, du,
\end{aligned}
$$

where $d\theta^\beta_{x,t,u}(\ )$ is the measure on the hypersurface

$$H_{x,t,u} = \{ z \in \mathbf{R}^n \colon \ z = utr(b, x, \overline{y})\overline{y}, \ \overline{y} \in \Omega_{n-1} \}$$

induced by $r^{n-2+|\beta|}(t,x,\overline{y}) p_\beta(\overline{y}) | \, d\sigma(\overline{y})$ on Ω_{n-1}.

Taking the Fourier transform $^{\wedge_y}$ we have

$$
\begin{aligned}
(1.3) \quad & \sum_{|\beta|\le -m} c(\alpha_0, m, \beta) \int_{\mathbf{R}^n} (-it\xi)^\beta \hat{f}(\xi) e^{i\xi\cdot x} \\
&\qquad \cdot \int_0^1 [d\mu^\beta_{x,t}]^\wedge (ut\xi) u^{|\beta|+2m+n-1} (1-u^2)^{\alpha_0-m-1} \, du \, d\xi,
\end{aligned}
$$

where $[d\mu^\beta_{x,t}]$ is the measure on $H_{x,t,1/t}$ induced by the above-mentioned measure on Ω_{n-1}. Let χ_1 be a cutoff function supported on $[-1,1]$ and identically one in $[-\frac{1}{2}, \frac{1}{2}]$, $\gamma_1 = |\beta| + 2m + n$,

$$
\begin{aligned}
(1.3) \quad &= \sum_{|\beta|\le -m} c(\alpha_0, m, \eta) \int (-it\xi)^\beta \hat{f}(\xi) \chi_1(t|\xi|) e^{ix\cdot\xi} \\
&\qquad \cdot \int_0^1 [d\mu^\beta_{x,t}]^\wedge (ut\xi) u^{\gamma-1} (1-u^2)^{\alpha_0-m-1} \, du \, d\xi \\
(1.4) \quad &+ \sum_{|\beta|\le -m} c(\alpha_0, m, \beta) \int (-it\xi)^\beta \hat{f}(\xi) (1 - \chi_1(t|\xi|)) e^{i\xi\cdot x} \\
&\qquad \cdot \int_0^1 [d\mu^\beta_{x,t}]^\wedge (ut\xi) u^{\gamma-1} (1-u^2)^{\alpha_0-m-1} \, du \, d\xi.
\end{aligned}
$$

The terms in the first sum, since the inner integral is a C^∞ function, can be written as a ΨdO whose symbol is in $S^{-\infty}_{1,0}$.

Let us split the integrals in the second sum, multiplying by $\chi_2(u)$ and $(1-\chi_2(u))$, where χ_2 is a cutoff function supported on $(0,2)$ and identically one in $[\frac{1}{2},\frac{3}{2}]$. Then

$$\int_0^1 [d\mu^\beta_{x,t}]^\wedge(ut\xi)u^{\gamma-1}(1-\chi_2(u))(1-u^2)^{\alpha_0-m-1}\,du$$

can be expressed with the notation of (1.2) as

$$\left\{\int_0^1 [d\theta^\beta_{x,t,u}(y)]u^{\gamma-1}(1-\chi_2(u))(1-u^2)^{\alpha_0-m-1}\,du\right\}^\wedge(\xi).$$

The integral is a function of ξ in $C^\infty(\mathbf{R}^n-\{0\})$ and C^γ at 0 supported on $A=\{z:|z|<tr(t,x,z/|z|)\}$. This is a consequence of the fact that $(1-u^2)^{\alpha_0-m-1}$ is a C^∞ bounded function in $\operatorname{supp}(1-\chi_2(u))$ and the family of hypersurfaces $\{H_{x,t,ut}\}_{u\in(0,1]}$ are C^∞ diffeomorphic to $A-\{0\}$. Hence, its Fourier transform is a symbol in $S^{-(n+|\beta|+2m)}_{1,0}$ and the corresponding terms in (1.3) are pseudodifferential operators with symbols in $S^{-(n+2m)}_{1,0}$.

It remains to study the leading terms involved in (1.3):

$$\int_0^1 [d\mu^\beta_{x,t}]^\wedge(ut\xi)u^{\gamma-1}\chi_2(u)(1-u^2)^{\alpha_0-m-1}\,du. \tag{1.5}$$

Claim. (1.5) may be written, modulo lower order terms, as

$$e^{itd(x,t,\xi)|\xi|}p_t(x,t\xi)+e^{-itd(x,t,\xi)|\xi|}q_t(x,t,\xi),$$

where p_t and q_t are symbols in $S^{-(n-1)/2-\alpha_0+m}_{1,0}(K_x\times\mathbf{R}^n_\xi)$ smoothly in $t\in[0,t_0)$ and $d(x,t;\xi)$ and smooth and homogeneous of degree zero in ξ.

According to [**4**], since $H_{x,t,1/t}$ satisfies the hypothesis there (i.e. nonzero curvature uniformly in t), we have the following asymptotic behavior:

$$[d\mu^\beta_{x,t}]^\wedge(\xi)=C_{\alpha_0}d(x,t,\xi)^{n-1}[\xi^{(n-1)/2}J_{(n-2)/2}(\varsigma)*g_\beta(x,t,\xi,(\cdot))^\wedge(\varsigma)],$$

where the convolution is one dimensional with respect to ς and is evaluated at the point $|\xi|d(x,t,\xi)$, d is the function in the claim, and $g_\beta(x,t,\xi,\nu)$ is in $C^\infty_0(\mathbf{R}_\nu)$, smooth and homogeneous of degree zero in ξ, and smoothly depending on x and t.

Since in (1.3) we have $1-\chi_1(t|\xi|)$, our study is reduced to the behavior of (1.4) when $|\xi|>1/2t$.

Hence (1.5) is the function

$$d(x,t,\xi)^{n-1}\int_0^1 \left(\varsigma^{-(n-2)/2}J_{(n-2)/2}(\varsigma)*g_\beta(x,t,\xi,(\cdot))^\wedge(\varsigma)\right|_{\varsigma=tu|\xi|d} \cdot u^{\gamma-1}\chi_2(u)(1-u^2)^{\alpha_0-m-1}\,du.$$

We can write this, for some $s\in S^{-(n-1)/2}_{1,0}$, as the sum of

$$d(x,t,\xi)^{n-1}\int_0^1 \left(s(\varsigma)e^{i\varsigma}*g_\beta(x,t,\xi(\cdot))^\wedge(\varsigma)\right)\right|_{\varsigma=tu|\xi|d} \cdot u^{\gamma-1}\chi_2(u)(1-^2)^{\alpha_0-m-1}\,du \tag{1.6}$$

and a similar conjugate expression.

We deal only with (1.6); the conjugate term can be treated in the same way.

Since d is homogeneous of degree zero, let us pay attention to

(1.7)
$$\int_0^1 e^{itu|\xi|d}(s(\varsigma) * g(x,t,\xi,(\cdot))^\wedge(\varsigma)e^{-i\varsigma})\Big|_{\varsigma=tu|\xi|d} u^{\gamma-1}(1-u^2)^{\alpha_0-m-1}\chi_2(u)\,du$$
$$= \mathrm{I}(x,t,t\xi).$$

We want to prove that $\mathrm{II}(x,t,t\xi) = e^{-it|\xi|d(x,t,\xi)}_{\mathrm{I}(x,t,t\xi)}$ is in the class $S_{1,0}^{-(n-1)/2-(\alpha_0-m)}$. Since g_β is homogeneous of degree zero, it suffices to estimate $\mathrm{II}(x,t,\xi)$ and its derivatives:

(1.8)
$$\text{(a)}\qquad |\mathrm{II}(x,t,\xi)| = \Big|\int_0^1 e^{iu|\xi|td}\int_{-\infty}^{\infty} g_\beta(x,t,\xi,(\cdot))^\wedge(y)e^{iy}s(u|\xi|d-y)\,dy$$
$$\cdot u^{\gamma-1}(1-u^2)^{\alpha_0-m-1}\chi_2(u)\,du\Big|$$
$$\le \int_{-\infty}^{\infty}\sup_{u_0\le u\le 1}|g_\beta(x,t,\xi,(\cdot))^\wedge s(u(|\xi|d-z))|$$
$$\cdot\Big|\int_J e^{iu(|\xi|d-z)}u^\gamma(1-u^2)^{\alpha_0-m-1}\,du\Big|\,dz$$

from the mean value theorem for integrals, where J is some subinterval of $[0,1]$.

REMARK.
$$\Big|\int_J e^{iu(|\xi|d-z)}u^\gamma(1-u^2)^{\alpha_0-m-1}\,du\Big|$$
is bounded uniformly (i.e. independently of J) by $c/(1+||\xi|d-z|^{\alpha_0-m})$ (see [**12**, p. 158]).

Hence, since $g_\beta(x,t,\xi,(\cdot))^\wedge(\eta)\in S^{-\infty}$, (1.8) is bounded by

$$C_{t,x}\int_{-\infty}^{\infty}\frac{C_p}{1+|u_0z|^p}\cdot\frac{1}{1+|u_0(|\xi|d-z)|^{(n-1)/2}}\frac{1}{1+||\xi|d-z|^{\alpha_0-m}}\,dz$$

for any integer p, and then the desired bound $c/(1+|\xi|^{(n-1)/2+\alpha_0-m})$ follows.

$$\text{(b)}\qquad \partial\mathrm{II}(x,t,\xi)/\partial\xi_1$$
$$= e^{i|\xi|d(x,t,\xi)}\Big\{-ib(x,t,\xi)\mathrm{I}(x,t,\xi)+\int_0^1 e^{iu|\xi|d}iub(x,t,\xi)$$
$$\cdot(s(\varsigma)*\hat g_\beta(\varsigma)e^{-i\varsigma})\,|_{\varsigma=u|\xi|d}|\,u^{\gamma-1}(1-u^2)^{\alpha_0-m-1}\chi_2(u)\,du$$
$$+\int_0^1 e^{iu|\xi|d}\left(\frac{\partial}{\partial\varsigma}s(\varsigma)*\hat g_\beta(\varsigma)e^{-i\xi}\right)\Big|_{\varsigma=u|\xi|d}$$
$$\cdot u^\gamma b(x,t,\xi)(1-u^2)^{\alpha_0-m-1}\chi_2(u)\,du$$
$$+\int_0^1 e^{iu|\xi|d}(s(\varsigma)*\frac{\partial}{\partial\xi_i}\hat g_\beta(x,t,\xi,(\cdot))(\varsigma)e^{-i\varsigma})\,|_{\varsigma=u|\xi|d}$$
$$\cdot u^{\gamma-1}(1-u^2)^{\alpha_0-m-1}\chi_2(u)\,du\Big\}$$

for $b(x,t,\xi) = (\partial/\partial\xi_i)(iu|\xi|d)$. The last two terms can be bounded by

$$\frac{c}{(1+|\xi|^{(n-1)/2+\alpha_0-m+1})}$$

as in (a), since $\partial s/\partial\varsigma \in S_{1,0}^{-(m-1)/2-1}$ and $(\partial\hat{g}/\partial\xi_1)(x,t,\xi,(\cdot))(\varsigma)$ is homogeneous of degree -1 in ξ and rapidly decreasing in ς, and $b(x,t,\xi)$ is homogeneous of degree zero in ξ.

We can put together the first two terms and write

$$-ie^{-i|\xi|d(x,t,\xi)}b(x,t\xi)\int e^{iu|\xi|d}(s(\varsigma)*\hat{g}(\varsigma)e^{-i\varsigma})\Big|_{\varsigma=u|\xi|d} \cdot u^{\gamma-1}(1-u^2)^{\alpha_0-m}\tilde{\chi}_2(u)\,du, \quad \text{where } \tilde{\chi}_2(u)=\chi_2(u)/(1+u),$$

which can be suitably bounded as was done in (a). Notice that higher derivatives w.r.t. ξ and x can be bounded in a similar way for $x\in K$.

Thus (1.4) can be written, modulo lower order terms, as

$$\iint p_t(x,t,\xi)e^{i(x-y)\cdot\xi+itd(x,t,\xi)|\xi|}[1-\chi_1(t|\xi|)]f(y)\,dy\,d\xi,$$

where $p_t\in S_{1,0}^{(1-n)/2-\alpha_0}(K_x\times R_\xi^n)$, uniformly in $t\in[0,t_0]$.

The function $(x-y)\cdot\xi+td(x,t,\xi)|\xi|$ has been proved in [**4**] to be a nondegenerate phase function for small t. This completes the proof of Lemma 11 in the case $\alpha=\alpha_0$.

We now prove Lemma 11 for general α. The leading term for T_t^α is

$$\begin{aligned}\frac{1}{\Gamma(\alpha-\alpha_0)}\operatorname{reg}\int_0^1\iint p_{tv}(x,tv\xi)e^{i(x-y)\cdot\xi+itvd(x,tv,\xi)|\xi|}[1-\chi_1(tv|\xi|)]f(y)\,dy\,d\xi \\ \cdot v^{n+1+2\alpha_0}(1-v^2)^{\alpha-\alpha_0-1}\,dv \\ =\iint e^{i(x-y)\xi}f(y)\left\{\frac{1}{\Gamma(\alpha-\alpha_0)}\operatorname{reg}\int_0^1 p_{tv}(x,tv\xi)e^{itvd(x,tv,\xi)|\xi|}\right. \\ \left.\cdot[1-\chi_1(tv|\xi|)]\;v^{n+1+2\alpha_0}(1-v^2)^{\alpha-\alpha_0-1}\,dv\right\}dy\,d\xi.\end{aligned}$$

The proof is reduced to showing that

$$\frac{1}{\Gamma(\alpha-\alpha_0)}\operatorname{reg}\int_0^1 p_{tv}(x,tv\xi)e^{itvd(x,tv,\xi)|\xi|} \cdot[1-\chi_1(tv|\xi|)]v^{n+1+2\alpha_0}(1-v^2)^{\alpha-\alpha_0-1}\,dv \tag{1.9}$$

can be written as

$$e^{itd(x,t,\xi)|\xi|}b_{t,\alpha_0}^\alpha(x,t\xi)$$

with the conditions of Lemma 11.

Let us remark that $b_{t,\alpha_0}^\alpha(x,t\xi)$ vanishes for $|\xi|<1/2t$ due to the properties of χ_1.

We will prove that $b_{t\alpha_0}^\alpha(x,t\xi)$ can be suitably bounded; the argument for higher derivatives is a straightforward repetition of the case $\alpha=\alpha_0$ above.

Assume $\alpha<\alpha_0$; then (1.9) is

$$\frac{1}{\Gamma(\alpha-\alpha_0)}\left[\int_0^1-1/2t|\xi|+\operatorname{reg}\int_{1-1/2t|\xi|}^1\right].$$

The first integral can be bounded by

$$\frac{c}{1+|t\xi|^{\alpha_0-(1-n)/2}}\int_0^{1-1/2t|\xi|}(1-v)^{\alpha-\alpha_0-1}\,dv=\frac{c}{1+|t\xi|^{\alpha_0-(1-n)/2}}\,\frac{1}{2t|\xi|^{\alpha_0-\alpha}}.$$

The second, from definition (A.2) (see Appendix 1), is

$$\frac{1}{\Gamma(\alpha-\alpha_0)}\int_{1-1/2t|\xi|}^{1}v^{n+1+2\alpha_0}(1-v^2)^{\alpha-\alpha_0-1}\left(\varphi(v)-\sum_{k=0}^{-[\alpha-\alpha_0]}\frac{\varphi^{(k)}(1)}{k!}(1-v)^k\right)dv$$
$$+\sum_{k=0}^{-\alpha}\varphi^{(k)}(1)c_k(\alpha,\alpha_0),$$

where

$$\varphi(v)=P_{tv}(x,tv\xi)e^{itvd(x,tv,\xi)|\xi|}(1-\chi_1(tv|\xi|)).$$

Notice that $v^{n+1+2\alpha_0}|\varphi^{(-[\alpha-\alpha_0]+1)}(v)|$ is bounded by

$$c(1+|t\xi|)^{(1-n)/2-\alpha_0-[\alpha-\alpha_0]+1};$$

hence the integral is bounded by

$$c(1+|t\xi|)^{(1-n)/2-\alpha_0-[\alpha-\alpha_0]+1}\int_{1-1/2t|\xi|}^{1}(1+v)^{\alpha-\alpha_0-1}(1-v)^{-[\alpha-\alpha_0]+\alpha-\alpha_0}\,dv,$$

which satisfies the appropriate bound.

LEMMA 12. *For a suitable function $c(t,x,\alpha)$, analytic in α and smooth in t and x, let us define*

$$g_\alpha(f)(x)^2=\int_0^{t_0}|T_t^\alpha f(x)=c(t,x,\alpha)T_t^1f(x)|^2\frac{dt}{t}.$$

Then

$$\|g_\alpha(f)\|_{L^2(K)}\le c_\alpha\|f\|_{L^2}$$

where c_α depends only on K and t_0 for $-(n-1)/2<\operatorname{Re}\alpha<0$.

PROOF. Let us consider F_t, the Egorov transform associated to ϕ_t in Lemma 11, i.e.

$$F_tf(x)=\int e^{iS_t(x,\xi)}\hat f(\xi)\,d\xi,\quad \text{where } S_t(x,\xi)=x\cdot\xi+td(x,t,\xi)|\xi|.$$

Modulo lower order terms, which are bounded by the leading one, we can write

$$T_t^\alpha f(x)=F_tb_t^\alpha(x,tD)f(x),$$

where $b_t^\alpha(x,tD)$ denotes the pseudodifferential operator whose symbol is $b^\alpha_{t,\alpha_0}(x,t\xi)$ in Lemma 11.

Let us take the wave packet transform W_g, which, according to [**1**], is a tranform $f(x)\to W_gf(x,\xi)$ defined by an "admissible section" $g(x,\xi)$, i.e. g is an elliptic matrix-valued symbol of order one such that $g(x,\xi)$ is in the Siegel upper half-space H^+. Let us restate some properties of W_g given in [**1**].

(1) Plancherel theorem:

$$\iint|W_gf(x,\xi)|^2\,dx\,d\xi=\int|f(x)|^2\,dx+\int|S^{-1/2}gf(x)|^2\,dx,$$

where $S_g^{-1/2}$ is a ΨdO of order $-\frac{1}{2}$.

(2) $W_g(p(x,D)f)(x,\xi) = p(x,\xi)W_g f(x,\xi)$ modulo lower order terms.

(3) For any Egorov transform F,

$$W_{\tilde{g}}Ff(x,\xi) = \mu(\tilde{x},\tilde{\xi},g)W_g f(\tilde{x},\tilde{\xi}),$$

where μ is an elliptic symbol of order zero, $(\tilde{x},\tilde{\xi}) = \Psi(x,\xi)$ is the canonical transformation generated by the phase $S(x,\xi)$,

$$\xi = \nabla_x S(x,\bar{\xi}), \quad \tilde{x} = \nabla_{\tilde{\xi}} S(x,\tilde{\xi}), \quad \text{and} \quad \tilde{g} = \begin{pmatrix} \partial\xi/\partial\tilde{\xi}, & \partial\xi/\partial\tilde{x} \\ \partial x/\partial\tilde{\xi}, & \partial x/\partial\tilde{x} \end{pmatrix} \circ g,$$

where "$\circ$" denotes the action of the symplectic group $S_{\mathrm{p}}(n,\mathbf{R})$ on H^+.

For $F_{t,u}$ the canonical transformation Ψ is given by

$$\tilde{x} = x_j + t\frac{\partial d}{\partial\tilde{\xi}_j}(x,t,\xi)|\xi| + td(x,t,\tilde{\xi})\frac{\tilde{\xi}_j}{|\tilde{\xi}|},$$

$$\xi_j = \tilde{\xi}_j + t\frac{\partial d}{\partial x_j}(x,t,\tilde{\xi})|\tilde{\xi}|.$$

As was proved in [**4**], for small t, $(x,\xi) \to (\tilde{x},\tilde{\xi})$ is a diffeomorphism homogeneous of degree one and the Jacobian of Ψ, $J_t(\tilde{x},\tilde{\xi})$, is an elliptic symbol of order zero.

For t_0 small enough, $t < t_0$, we can find a matrix $\tilde{g}_t$ such that

$$(3') \qquad W_{\tilde{g}_t}F_t f(x,\xi) = \mu(\tilde{x},t,\tilde{\xi},g)W_g f(\tilde{x},\tilde{\xi}),$$

where g is independent of t.

Then by (1), modulo lower order terms,

$$\begin{aligned}
&\int g_\alpha f(x)^2\,dx \\
&= \int_0^{t_0}\left\{\int |F_t b_t^\alpha(x,tD)f(x) - c(x,t,\alpha)F_t b_t^1(x,tD)f(x)|^2\,dx\right\}\frac{dt}{t} \\
&= \int_0^{t_0}\iint |W_{\tilde{g}t}F_t b_t^\alpha(x,tD)f(x,\xi) - W_{\tilde{g}t}c(x,t,\alpha)F_t b_t^{-1}(x,tD)f(x,\xi)|^2\,dx\,d\xi\frac{dt}{t} \\
&= \int_0^{t_0}\iint |\mu(\tilde{x},t,\tilde{\xi},g)(W_g b_t^\alpha(x,tD)f(\tilde{x},\tilde{\xi}) \\
&\qquad\qquad - c(x,t,\alpha)W_g b_t^1(x,tD)f(\tilde{x},\tilde{\xi}))|^2\,dx\,d\xi\,\frac{dt}{t} \\
&= \int_0^{t_0}\iint |\mu(\tilde{x},t,\tilde{\xi},g)(b_t^\alpha(\tilde{x},t\tilde{\xi})W_g f(\tilde{x},\tilde{\xi}) \\
&\qquad\qquad - c(x,t,\alpha)b_t^1(\tilde{,}xt\tilde{\xi})W_g f(\tilde{x},\tilde{\xi}))|^2\,dx\,d\xi\,\frac{dt}{t} \\
&= \iint |W_g f(\tilde{x},\tilde{\xi})|^2\int_0^{t_0}|\mu(\tilde{x},t,\tilde{\xi},g)(b_t^\alpha(\tilde{x},t\tilde{\xi}) - \tilde{c}(\tilde{x},\tilde{\xi},t,\alpha)b_t^1(\tilde{x},t\tilde{\xi}))|^2 \\
&\qquad\qquad \cdot |J_t(\tilde{x},\tilde{\xi})|\frac{dt}{t}\,d\tilde{x}\,d\tilde{\xi}.
\end{aligned}$$

The inner integral is bounded uniformly in $\tilde{x}$ and $\tilde{\xi}$, as shown in [**4**], by splitting it into

$$\int_0^{1/4|\xi|} + \int_{1/4|\xi|}^{t_0}$$

since $b_t^\alpha(\tilde{x}, t\tilde{\xi})$ vanishes when $t < 1/2|\xi|$. We have to consider the first integral only for the term in the expression of $T_t^\alpha f(x)$ given by the analytic continuation of the first sum in (1.4). $c(t, x, \alpha)$ above is chosen so that it kills the singularity at the origin of this integral. It is easy to see that $c(t, x, \alpha)$ is an analytic function of . The second integral is bounded for $-(n-1)/2 < \operatorname{Re}\alpha < 0$. Then, since J and μ are symbols of order zero, our statement follows from properties (1) and (2).

LEMMA 13.

$$\left\| \sup_{0<t<t_0} \left(\frac{1}{t} \int_0^t |T_u^\alpha f(x)|^2 \, du \right)^{1/2} \right\|_{L^2(K)} \leq c\|f\|_{L^2}$$

for $-(n-1)/2 < \operatorname{Re}\alpha < 0$.

PROOF.

$$\begin{aligned}
\frac{1}{t} \int_0^t |T_u^\alpha f(x)|^2 \, du \frac{1}{t} \int_0^t |T_u^\alpha f(x) \\
&= c(x, u, \alpha) T_u^1 f(x)|^2 \, du + \frac{1}{t} \int_0^t |c(x, u, \alpha) T_u^1 f(x)|^2 \, du \\
&\leq \int_0^t |T_u^\alpha f(x) - c(x, u, \alpha) T_u^1 f(x)|^2 \frac{du}{u} + \bar{c}(x, \alpha)^2 \sup_{0<t<t_0} |T_t^1 f(x)|^2 \\
&\leq g_\alpha(f)(x)^2 + \bar{c}(x, \alpha) \sup_{0<t<t_0} |T_t^1 f(x)|^2,
\end{aligned}$$

where $\bar{c}(x, \alpha) = \sup_{0<t<t_0, x \in K} c(x, t, \alpha)$.

LEMMA 14. *Let* $\alpha_0 > -n/2$ *and* $1 < p < \infty$. *Then for any sufficiently small* $\varepsilon > 0$,

$$\left\| \sup_{0<t<t_0} |T_t^\gamma f| \right\|_{L^p(K)} \leq C_{p,K,\varepsilon} \|f\|_{L^p} \quad \textit{when } \operatorname{Re}\gamma = 1 + \varepsilon.$$

PROOF.

$$\begin{aligned}
T_t^\gamma f(x) &= \frac{2}{\Gamma(1 - \alpha_0 + \varepsilon)} \int_0^1 M_{tv,1,j}^{\alpha_0} f(x) v^{n-1+2\alpha_0} (1 - v^2)^{-\alpha_0 + \gamma - 1} \, dv \\
&= \frac{2}{\Gamma(i \operatorname{Im}\alpha_0 + \varepsilon/2)(1 - \alpha_0 + \varepsilon)} \\
&\quad \cdot \int_0^1 \int_0^1 M_{tv,u}^{\operatorname{Re}\alpha_0 - \varepsilon/2} f(x) v^{n-1+2\alpha_0} u^{n-1+\operatorname{Re}\alpha_0 - \varepsilon} (1 - v^2)^{-\alpha_0 + \gamma - 1} \\
&\qquad \cdot (1 - u^2)^{i \operatorname{Im}\alpha + \varepsilon/2 - 1} \, du \, dv.
\end{aligned}$$

The result for general α follows from the case $\operatorname{Re}\alpha_0 - \varepsilon/2$ integer by using complex interpolation (reducing the α_0's to $\alpha_0 > -n/2$). So let us assume $\operatorname{Re}\alpha_0 - \varepsilon/2 = m$

integer. In this situation the above expression is

$$c(\alpha_0)\int_0^1\int_0^1 \left\langle (tv)^{-2m-n+2}\delta^{(m)}\left((tvu)^2-\frac{|y|^2}{r^2(tv,x,\overline{y})}\right)\{f(x-y)a(tv,x-y)\}\right\rangle \cdot u^{\text{pow}}\, v^{\text{pow}}\cdot(1-v^2)^{-\alpha_0+\gamma-1}(1-u^2)^{i\operatorname{Im}\alpha+\varepsilon/2}\,du\,dv,$$

as can be seen by following the lines of Lemma 11. The expression of $\delta^{(m)}$ gives us (a is a C^∞ function and can be erased)

$$\sum_{q=0}^{-m} ct^q\int_0^1\int_0^1 [d\theta^q_{x,tv,u}]\left(\frac{\partial^q f(x-s\overline{y})}{\partial s^q}\right) \cdot v^{\text{pow}}\, u^{\text{pow}}(1-v^2)^{-\alpha_0+\gamma-1}(1-u^2)^{i\operatorname{Im}\alpha+\varepsilon/2-1}\,du\,dv,$$

where $d\theta^q_{x,tv,u}$ is the measure carried on the hypersurface

$$H_{x,tv,u}=\{z\in\mathbf{R}^n : z=tvur(tv,x,\overline{y})\overline{y},\ \overline{y}\in\Omega_{h-1}\}$$

defined in (1.2) of Lemma 11. From the formula in the proof of Lemma 8′, (1.10) becomes

$$=\sum_{q=0}^{-m} ct^q\int_0^1\int_0^1\int_v^1 [d\theta^q_{x,tv,u}]\left(\frac{\partial^q f(x-s\overline{y})}{\partial s^q}\right) \cdot(1-w^2)^{-\alpha_0+\gamma-2}wv^{\text{pow}}\,u^{\text{pow}}(1-u^2)^{i\operatorname{Im}\alpha+\varepsilon/2-1}\,dw\,du\,dv \tag{1.11}$$

$$=\sum_{q=0}^{-m} ct^q\int_0^1\int_0^1\int_1^w [d\theta^q_{x,tv,u}]\left(\frac{\partial^q f(x-s\overline{y})}{\partial s^q}\right)v^{n-1+q+2_0}\,dv \cdot(1-w^2)^{-\alpha_0+\gamma-2}w\,dw\,u^{\text{pow}}\,(1-u^2)^{i\operatorname{Im}\alpha+\varepsilon/2-1}\,du. \tag{1.12}$$

Due to the properties of $H_{x,tv,u}$, the inner integral is

$$\int_{V_{x,t,w,u}}\psi_{x,t,w,u}(y)\frac{\partial^q f(x-y)}{\partial s^q}\,dy, \tag{1.13}$$

where $V_{x,t,w,u}$ is the region in $\mathbf{R}^n$ defined by

$$\{y:\ |y|\le utvr(tv,x,\overline{y}) \text{ for some } 0<v<w\}.$$

$\psi_{x,t,w,u}$ is smooth w.r.t. x,t,w,u,y and is uniformly bounded in x,eK,t,u and w.

The family of solids $\{V_{x,b,w,u}\}_{C<t<t_0,0<w<1,\ 0<u<1}$ is regular with respect to cubes (in the sense of the theory of differentiation of integrals); hence we can expect to bound the integral (1.13) by the maximal Hardy-Littlewood function $f^*(x)$ for x in compact sets. This is straightforward for $q=0$ (the only value of q when $\operatorname{Re}\alpha_0-\varepsilon/2=0$) and (1.12) is bounded for $\operatorname{Re}-\varepsilon/2=0$ by

$$c\int_0^1\int_0^1 |f^*(x)|(1-w^2)^{\varepsilon/2-1}(1-u^2)^{\varepsilon/2-1}\,du\,dw.$$

The statement is proved in this case and can be done very similarly for the integral in (1.12) with $q=0$.

In the case $q \neq 0$, $\operatorname{Re}\alpha_0 - \varepsilon/2 = m < 0$, we use an inductive argument:

$$\begin{aligned} ct^q \int_0^1 \int_0^1 \int_{V_{x,t,w,u}} & \Psi_{x,t,w,u}(y) \frac{\partial^q f(x - s\overline{y})}{\partial s^q}\, dy \\ & \cdot (1-w^2)^{-\alpha_0+\gamma-2} w\, dw\, u^{\mathrm{pow}}(1-u^2)^{i \operatorname{Im}\alpha+\varepsilon/2-1}\, du \\ = ct^q \int_0^1 \int_0^1 \int_V & \cdot\overline{y} \cdot \nabla y \left(\frac{\partial^{q-1} f(x - s\overline{y})}{\partial s^{q-1}} \right) dy (1-w^2)^{-\alpha_0+\gamma-2} w\, dw \\ & \cdot u^{\mathrm{pow}}(1-u^2)^{i \operatorname{Im}\alpha+\varepsilon/2-1}\, du \\ = ct^q \int_0^1 \int_0^1 \int_V & \operatorname{div} \left(\frac{\partial^{q-1} f(x - sy)}{\partial s^{q-1}} \psi \overline{y}_j \right) dy (1-w^2)^{-\alpha_0+\gamma-2} w\, dw \\ & \cdot u^{\mathrm{pow}}(1-u^2)^{i \operatorname{Im}\alpha+\varepsilon/2-1}\, du \\ - ct^q \int_0^1 \int_0^1 \int_V & \frac{\partial^{q-1} f}{\partial s^{q-1}}(x - y) \operatorname{div}[\psi_{x,t,w,u}(y) \cdot \overline{y}_j]\, dy \\ & \cdot (1-w^2)^{-\alpha_0+\gamma-2} w\, dw\, u^{\mathrm{pow}}(1-u^2)^{i \operatorname{Im}\alpha+\varepsilon/2-1}\, du. \end{aligned}$$

The last integral can be bounded by $cf^*(x)$ by induction hypothesis, and the first, by the divergence theorem, is

$$\begin{aligned} (1.14)\quad ct^q \int_0^1 \int_0^1 \int_{\partial V_{x,t,w,u}} & \frac{\partial^{q-1} f(x - s\overline{y})}{\partial s^{q-1}} (x-y) \left(\sum \psi_{x,t,w,u}(y) y_j \eta_j \right) dy \\ & \cdot (1-w^2)^{-\alpha_0+\gamma-2} w\, dw u^{\mathrm{pow}}(1-u^2)^{i \operatorname{Im}\alpha+\varepsilon/2-1}\, du, \end{aligned}$$

where $\eta = (\eta_j)$ is the normal to the hypersurface $\partial V_{x,t,w,u}$.

The inner integral in (1.14) can be reduced to a volume integral similar to (1.13) with $\partial^{q-1}/\partial s^{q-1}$ by once again using the formula in Lemma 8′ for $(1-w^2)^{-\alpha_0+\gamma-2}$ and changing the order of integration.

LEMMA 15. *Let $f \in S$, $\operatorname{Re}\alpha > 1/2 - (n-1)/2$. Then*

$$\left\| \sup_{0<t<t_0} |T_t^\alpha f(x)| \right\|_{L^2(K)} \leq c_\alpha \|f\|_{L^2}.$$

The lemma is an easy consequence of Lemmas 8′ and 13 and Hölder's inequality.

PROOF OF THEOREM 10. For $\operatorname{Re}\alpha = 0$, the operator T_j^0 is bounded $L^\infty \to L^\infty(K)$ (which can be proved as in Lemma 14).

Let us linearize $T_j^\alpha(x)$ (stopping time process); then, since all the constants in the above lemmas are at most exponentially increasing w.r.t. α complex interpolation applies between

$$\operatorname{Re}\alpha_1 > 1/2 - (n-1)/2, \quad p_1 = 2, \qquad \text{and} \qquad \operatorname{Re}\alpha_2 = 1 + \varepsilon, \quad 1 < p_2 < \infty,$$

and we obtain T_j^α boundedness for $n/(\alpha_n - 1) < p_\alpha < n/(1-\alpha)$. Interpolation between

$$\operatorname{Re}\alpha_1 > 1/2 - (n-1)/2, \quad p_1 = 2, \qquad \text{and} \qquad \operatorname{Re}\alpha_2 = 0, \quad p_2 = \infty,$$

shows T_j^α is bounded for $p_\alpha = (n-2)/2 - \varepsilon$ and $1/2 - (n-1)/2 < \alpha < 0$, $\varepsilon > 0$. The same intervals of boundedness hold for M_j^α.

Let S be a spacelike surface for L and consider γ: $S \times [0, t_0] \to \mathbf{R}^{n+1}$ smooth such that for every $y \in S$, $\gamma(y, 0) = y$ and $\gamma(y,t)$ is a timelike curve.

THEOREM 16. *Let $f \in L^p_{\mathrm{loc}}(S)$, $2n/(n+1) < p < 2(n-2)/(n-3)$, and let $u(x)$ be the solution of problem (1′). Then $u(\gamma(y,s))/s \to c(y)f(y)$ a.e. $y \in S$ when $s \to 0$, where $c(y)$ is $C^\infty(S)$ and independent of f.*

PROOF. By Lemma 3 we reduce problem (1′) to (1), where S is $t \equiv 0$; furthermore, we can choose x and t such that γ switches to $(\eta(x,t),t)$, where $\eta\colon \mathbf{R}^n \times [0,t_0] \to \mathbf{R}^n$, $\eta(x,0)=x$. By Remark 9,

$$u(\eta(x,t),t) = \sum_{j=0}^{N} t^{j+1} M_{t,1,j}^{-(n-3)/2+j} f(\eta(x,t)) + \frac{1}{t^{n-1}} \int_{\mathbf{R}^n} R_N(t,\eta(x,t),z) f(z)\, dz.$$

Since γ is timelike we may write $M^\alpha_{t,1,j} f(\eta(x,t))$ in the same form as $M^\alpha_{t,1,j} f(x)$ and get L^p-boundedness for the same p's.

Let us apply to $u(\eta(x,t),t)/t$ the classical argument involving L^p-norms and maximal operators to prove a.e. convergence. The worst maximal operator in the above expression for u is $M_j^{-(n-3)/2}$, which is bounded in L^p for $2n/(n+1) < p < 2(n-2)/(n-3)$. The remainder,

$$\sup_{0<t<t_0} \frac{1}{t^n} \int_{\mathbf{R}^n} |R_N(t,\eta(x,t),z) f(x)|\, dz,$$

is bounded by the strong maximal function.

The above argument proves that

$$\left\| \sup_{0<t<t_0} \frac{u(\gamma(y,s))}{s} \right\|_{L^p(K)} \le c_p \|f\|_{L^p(S)} \quad \text{for } f \in L^p(S).$$

Since the theorem is true for $f \in C^\infty$, a limiting argument proves our statement.

Appendix 1. *An analytic extension.* We define

$$\operatorname{reg} \int_0^1 \varphi(v) v^\beta (1-v^2)^{\alpha-1}\, dv \quad \text{for } \operatorname{Re}\alpha > -m,\ \operatorname{Re}\beta > 0$$

and φ a real C^∞ function, following [**3**, §3.8].

We may write ($\operatorname{Re}\alpha > 0$)

$$\begin{aligned}\int_0^1 \varphi(u) v^\beta (1-v^2)^{\alpha-1}\, dv = {} & \int_0^1 v^\beta (1-v^2)^{\alpha-1} \left[\varphi(v) - \sum_{k=0}^{m-1} \frac{\varphi^{(k)}(1)}{k!} (1-v)^k \right] dv \\ & + \sum_{k=0}^{m-1} \int_0^1 v^\beta (1+v)^{\alpha-1} (1-v)^{\alpha+k-1}\, dv \frac{\varphi^{(k)}(1)}{k!}.\end{aligned}$$

The first integral is convergent for $\operatorname{Re}\alpha > -m$, so we have to extend the second integral, which can be expressed as

$$\begin{aligned}&\int_0^1 (1-v)^{k+\alpha-1} \left\{ v^\beta (1+v)^{\alpha-1} - \sum_{j=0}^{m-1} \left[\sum_{p+q=j} \frac{\Gamma(\beta)\Gamma(\alpha) 2^{\alpha-q-1}}{\Gamma(\beta-p)\Gamma(\alpha-q)} \right] \frac{(1-v)^j}{j!} \right\} dv \\ &\quad + \sum_{j=1}^{m-1} \sum_{p+q=j} \frac{\Gamma(\beta)\Gamma(\alpha) \prod_{l\ge 0}^{k+j-1+q} (\alpha-q+l) 2^{\alpha-q-1}}{\Gamma(\beta-p)\Gamma(\alpha+k+j+1) j!}.\end{aligned}$$

This is well defined for any α, $\operatorname{Re}\alpha$ noninteger, $\operatorname{Re}\alpha > -m$, and can be written as $\Gamma(\alpha)C_k(\beta,\alpha)$, where $C_k(\beta,\alpha)$ is analytic in α for $\operatorname{Re}\alpha > -m$. Then:

DEFINITION. For $\beta > 0$, $\alpha > -m$, $\varphi \in C^\infty(\mathbf{R})$,

$$\begin{aligned} &\operatorname{reg}\int_0^1 \varphi(v)v^\beta(1-v^2)^{\alpha-1}\,dv \\ \text{(A.1)}\qquad &= \int_0^1 v^\beta(1-v^2)^{\alpha-1}\left[\varphi(v) - \sum_{k=0}^{m-1}\frac{\varphi^k(1)}{k!}(1-v)^k\right]dv \\ &+ \sum_{k=0}^{m-1}\Gamma(\alpha)C_k(\beta,\alpha)\frac{\varphi^{(k)}(1)}{k!}. \end{aligned}$$

DEFINITION. For $\beta > 0$, $\operatorname{Re}\alpha > -m$ and $\varphi \in C^\infty(\mathbf{R})$, we have the following analytic function of α:

$$\begin{aligned} &\frac{\operatorname{reg}}{\Gamma(\alpha)}\int_0^1 \varphi(v)v^\beta(1-v^2)^{\alpha-1}\,dv \\ \text{(A.2)}\qquad &= \frac{1}{\Gamma(\alpha)}\int_0^1 v^\beta(1-v^2)^{\alpha-1}\left[\varphi(v) - \sum_{k=0}^{m-1}\frac{\varphi^{(k)}(1)}{k!}(1-v)^k\right]dv \\ &+ \sum_{k=0}^{m-1}C_k(\beta,\alpha)\frac{\varphi^{(k)}(1)}{k!}. \end{aligned}$$

REMARK.

$$\text{(A.3)}\qquad \frac{\operatorname{reg}}{\Gamma(\alpha)}\int_0^1 \varphi(v)v^\beta(1-v^2)^{\alpha-1}\,dv\bigg|_{\alpha=0} = \frac{1}{2}\varphi(1).$$

Appendix 2. *Some results on hyperbolic PDEs.* We summarize some concepts and facts used in the last section; nevertheless, we omit some proofs and merely outline others; this is the case for Theorem 5, in Ludwig's works [**8**, **9**]. We select facts from both papers and point out some properties not explicitly stated in them, but necessary in our work. For unproved results, see Courant and Hilbert [**2**, Chapter VI, Part I].

DEFINITION. The "normal cone" at $(t_0, x_0) \in D$ for the strongly hyperbolic equation (0) is the set of $(\xi_0, \ldots, \xi_n) \in \mathbf{R}^{n+1}$ satisfying the algebraic "characteristic equation" at (t_0, x_0), i.e.

$$\sum_{i,j=0}^{n} a_{ij}(t_0, x_0)\xi_i\xi_j = 0.$$

The first order PDE,

$$\text{(2}'\text{)}\qquad \sum_{i,j=0}^{n} a_{ij}(x_0, x_1, \ldots, x_n)\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j} = 0, \quad \text{where } x_0 = t,$$

is called the Eiconal equation.

DEFINITION. If $\phi(x_0, \ldots, x_n)$ is a solution of (2′) the n-manifold $\phi(t, x) =$ constant is called "a characteristic surface" for L.

The characteristic surface $\phi(t,x) = c$, whose intersection with the initial surface $t = 0$ is an $(n-1)$-plane, is called a "planelike surface", i.e. $\phi(0,x) = c$ represents a plane $\omega \cdot x = \omega \cdot x_0$, where $\omega \in \Omega_{n-1}$.

Let us note some facts about solutions of (2):

A path $x_0(s), x_1(s), \ldots, x_n(s), p_0(s), \ldots, p_n(s)$ is called a "ray" or "bicharacteristic strip" when it is a solution of the canonical system of ODEs

$$\begin{aligned} \dot{x}_i &= Q_{pi}, \\ \dot{p}_i &= -Q_{xi}, \end{aligned} \qquad i = 0, \ldots, n, \tag{3}$$

and $Q(\overline{x}(s), \overline{p}(s)) = 0$, where $x_0 = t$, and

$$Q(\overline{x}, \overline{p}) = \sum_{i,j=0}^{n} a_{ij}(\overline{x}) p_i p_j,$$

and denotes derivative with respect to s.

DEFINITION. The set of points in a neighborhood of $(x_0^0, x_1^0, \ldots, x_n^0)$ which are x-spaces projections of all rays through the point $(x_0^0, \overline{x}^0)$ is the "ray conoid" centered at $(x_0^0, \overline{x}^0)$ associated with the operator L.

LEMMA. *Any characteristic surface $\overline{\phi}$ is locally generated by rays, i.e. in a neighborhood of a "suitable $(n-1)$-manifold strip" contained in $\overline{\phi}$, $\overline{\phi}$ is composed of all rays through the "manifold strip".*

LEMMA. *The ray conoid centered at $(t^0, \overline{x}^0)$ is locally the envelope of the planelike surfaces through $(t^0, \overline{x}^0)$.*

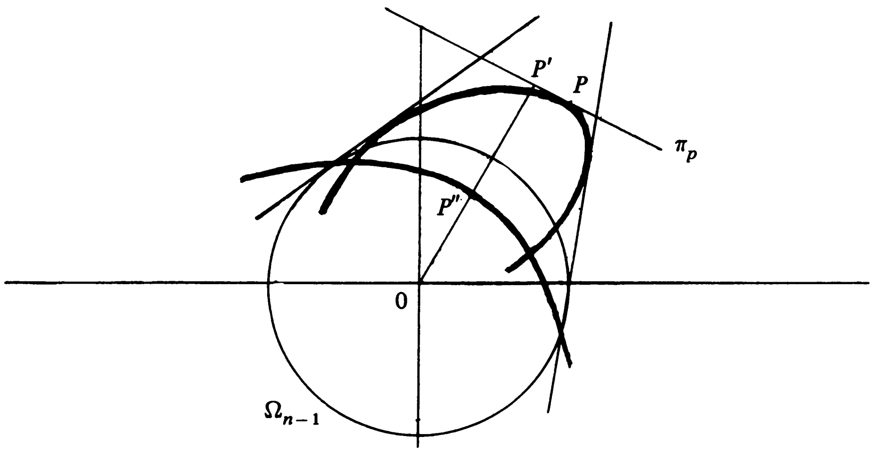
The envelope of a family of surfaces $\phi(t, \overline{x}; \lambda) = 0$, where λ is a vector-valued parameter, is the surface $\phi(t, \overline{x}; \lambda(t, \overline{x})) = 0$, where $\lambda(t,x)$ are the values of λ satisfying $\overline{x}\phi_\lambda(t, \overline{x}; \lambda) = 0$.

LEMMA. *Two distinct characteristic surfaces with a common interior point intersect along a ray.*

Since, in the constant coefficient case, planelike surfaces are n-dimensional affine submanifolds and rays are straight lines, it makes sense to introduce the "ray surface" and "normal surface". The ray surface is the intersection of a ray conoid centered at a point (in this case the ray cone) with the plane $t = 1$; it is a hypersurface in the -dimensional x-space. The normal surface is the intersection of a normal cone with the plane $\xi_0 = -1$, namely $G(-1, \xi_1, \ldots, \xi_n) = 0$. The following two lemmas hold in the constant coefficients case.

LEMMA. *Normal and ray surfaces are dual in the following sense: to any point P on the normal surface there corresponds a point P'' on the ray surface, obtained by inverting P' with respect to the unit sphere centered at the origin, where P' is the intersection of the tangent plane x_p to the normal surface at P and the normal line to π_p through the origin (see figure).*

For strongly hyperbolic equations of second order with constant coefficients, the normal surface is an ellipsoid, and both normal and ray surfaces are composed of



FIGURE

only one sheet. This, and the fact that dual surfaces of quadratic surfaces are also quadratic surfaces, and some other properties of convex hulls of these surfaces, cusp points and conical points, imply the following lemma.

LEMMA. *Ray and normal surfaces have nonzero curvature everywhere.*

We can extend the lemma to variable C^∞ coefficients, taking into account that (3) has C^∞ coefficients and the theorem on continuous and differentiable dependence on parameters for ODE applies here.

LEMMA A1. *Let us consider L (under conditions in (0)). Then the intersection of the ray conoid at $(0, \overline{x}^0)$ with the plane $t = \varepsilon$ has nonzero curvature everywhere for any ε sufficiently small.*

REMARK. In other words the concepts introduced in the case of constant coefficients are also locally meaningful for strongly hyperbolic equations with variable coefficients.

SKETCH OF PROOF OF THEOREM 5. Let us take the plane wave expansion for $\delta(x-z)$ (see [**2**]),

$$\delta(x-z) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}} \int_{\Omega_{n-1}} \delta^{(n-1)}(\omega(x-z))\, d\omega \qquad (n \text{ odd}), \tag{4}$$

and consider the solution $R_\omega(t,x,z)$ of the problem

$$Lu = 0, \quad u(0,x) = 0, \quad \frac{\partial u}{\partial t}(0,x) = \delta^{(n-1)}(\omega(x-z)), \tag{5}$$

where $\omega \in \Omega_{n-1}$. Then

$$R(t,x,z) = \text{constant} \int_{\omega \in \Omega n-1} R_\omega(t,x,z)\, d\omega. \tag{A.4}$$

According to [**8**, §5], we get the progressive generalized wave solution for (5),

$$\begin{aligned} R_\omega(t,x;z) =& \sum_{j=0}^{N} \delta^{(n-2-j)}(\overline{\phi}(t,x;z,\omega))\overline{\overline{a}}_j(t,x;z,\omega) + \overline{R}_N(t,x;z,\omega) \\ =& u^N + \overline{R}_N, \end{aligned}$$

where $\overline{\phi}$ is a defining function for a planelike characteristic surface associated to equation (2) with initial data

$$(*) \qquad \overline{\phi}(0,x;z,\omega) = 0 \quad \text{and} \quad \text{the hyperplane } \omega(x-z) = 0,$$

and the $\overline{\overline{a}}_j$'s are determined by solving a linear first order system ODE along the bicharacteristics which generate the surface $\phi =$ constant. Thus they satisfy the conditions of a_j in Theorem 5.

R_N can be made as smooth as desired by taking N large enough.

We will apply the stationary phase method (Lemma 5.2 in [**9**]) to

$$\text{(A.5)} \qquad \begin{aligned} R(t,x;z) =& \sum_{j=0}^{N} \int_{\Omega_{n-1}} \delta^{(n-2+j)}(\overline{\phi}(t,x;z\omega))a_j(t,x;z,\omega)\,d\omega \\ &+ \int_{\Omega_{n-1}} R(t,x;z,\omega)\,d\omega. \end{aligned}$$

LEMMA A2. *Let $\xi = (\xi^1,\dots,\xi^k)$ be a vector-valued parameter, $F(s)$ a distribution smooth everywhere except the origin, and $\phi(x,\xi)$ a C^∞ function of x and ξ. Assume ξ is determined as a function $\xi(x)$ by the condition of stationary phase, $\phi_\xi(x,\xi) = 0$. Suppose*

$$\phi(x,\xi) = \phi(x,\varsigma(x)) + \sum_{i=1}^{k}(\xi_i - \varsigma_i(x))^2\Psi i(x,\xi),$$

where $\Psi_i(x,\varsigma(x)) = \phi_{\xi^i\xi^i}(x,\varsigma(x))$ satisfies

$$\begin{aligned} &\phi_{\xi^i\xi^i}(x,\varsigma(x)) > 0, \qquad i = 1,\dots,k, \\ &\phi_{\xi^i\xi^i}(x,\varsigma(x)) = 0, \qquad i \neq j. \end{aligned}$$

Define

$$J(x) \equiv \int F(\phi(x,\xi))a(x,\xi)\,d\xi,$$

where a is a C^∞ function with compact support in ξ. Then J an be written as

$$\sum_{j=0}^{N}(I_+^{(j+1)/2}F)(\phi(x,\varsigma(x)))\tilde{a}_j(x) + h^n(x),$$

where

$$\tilde{a}_j = \frac{c_j a_j(x,\varsigma(x))}{|\pi\phi_{\xi^i\xi^i}(x,\varsigma(x))|^{1/2}}$$

where a_j's C^∞ functions, $I_+^\alpha F(S)$ is the fractional integral of F, i.e.

$$I_+^\alpha F(S) = \frac{1}{\Gamma(\alpha)}\int_{-A}^{s} F(t)(s-t)^{\alpha-1}\,dt \quad \textit{for an } A > 0,$$

and $h^N(x)$ can be made as smooth as desired by taking N large enough.

Let us continue with the proof of Theorem 5. At each point $w \in \Omega_{n-1}$, there exist a neighborhood U_ω and a change of coordinates to $\overline{\omega}^j$ such that Hessian $(n-1)$-matrix of $\overline{\phi}$ (i.e., ϕ with respect to the new coordinates),

$$\overline{\phi}_{\overline{\omega}^i\overline{\omega}^j}(t, x; z, \varsigma(t, x, z)),$$

is diagonal for $\varsigma(t, x, z) \in U_\omega$ (a consequence of Morse's Lemma). Thus if we prove $\overline{\phi}_{\overline{\omega}^i\overline{\omega}^i}(t, x, z, \varsigma) > 0$, by using a partition of unity, our function $\overline{\phi}$ satisfies the hypothesis of the above lemma.

According to [**9**, §6.D], $\overline{\phi}_{\overline{\omega}^i\overline{\omega}^i}(t, x, z, \varsigma)$ is the difference between the curvature for t constant at (t, x) of the ray conoid centered at $(0, z)$ and the curvature of the planelike surface tangent to the ray conoid at (t, x) when t is constant, where the curvatures measures along the curve $\overline{\omega}^i =$ constant.

This means that for t small enough, since planelike surfaces approach planes, we can make

$$\operatorname{sign} \tilde{\phi}_{\overline{\omega}^i\overline{\omega}^i}(t, x, z, \varsigma) = \operatorname{sign} K_i,$$

where K_i is one of the principal curvatures of the intersection of the ray conoid and the plane $t =$ constant. By Lemma A1 we are done.

Therefore each term in (A.5) is

(A.6)

$$T_j(t, x, z) = \sum_{i=0}^{M_j} I_+^{(n-1)/2+i} \delta^{n-2-j}(\overline{\phi}(t, x; z, \varsigma(t, x, z)))\tilde{a}_{ij}(t, x, z) + \tilde{h}_{M_j}(t, x, z).$$

Also, by [**9**],

$$\left[\frac{1}{\pi\overline{\phi}_{\overline{\omega}_i\overline{\omega}_i}(t, x, z, \varsigma)}\right]^{1/2} = t^{-(n-1)/2} b(t, x; z),$$

where $b(t, x, z)$ is a C^∞ function for $x, t \in \mathbf{R}^n$, $t \in [0, t_0]$, so by Lemma A2, $\tilde{A}_{ij}$ can be written as

$$a_{ij}(t, x, z)t^{-(n-1)/2}.$$

Let us add all the terms h_{M_j} in (A.6) and $\int_{\Omega_{n-1}} R_N(t, x, z, \omega)\, d\omega$ and denote the sum $R_N(t, x, z)$ by

$$R(t, x, z) = u^N + R_N(t, x, z).$$

By taking M_j and N large enough we can ensure that Lu^N has derivatives in the strict sense up to a certain order. Then R_N must be the solution of

$$Lv = -L|u^N, \qquad v(0, y) = \partial v(0, y)/\partial t = 0.$$

Since L is supported in the inside of the ray conoid according to the standard existence and uniqueness theorems (see [**6**]), we claim v exists, is uniquely determined, and has as many derivatives as we want by taking N and M_j large enough. v satisfies the condition on the support stated in Theorem 5.

As we pointed out $\phi = \overline{\phi}(t, x, z, \varsigma(t, x, z))$ is the regular envelope of the ω-parametric family $\overline{\phi}(t, x, z, \omega)$. Thus it is the ray conoid with vertex at $(0, z)$.

From Lemma A1 the ray conoid is a regular surface which can be written

$$p(t, x, z)\left(t - \frac{|z - x|}{r(t, z, (x - z)/|x - z|)}\right) = 0,$$

where $p(t,x,z) \neq 0$, and $r(t,x,(x-z)/|x-z|)$ satisfies the conditions of Theorem 5. Hence we may include $p^{\beta}(t,x,z)$ for some β in $a_{ij}(t,x,z)$ and consider

$$\phi\left(t - \frac{|z-x|}{r(t,z,(z-x)/|z-x|)}\right)$$

as the argument of the distribution.

Since $\partial\phi/\partial s = 1/r(t,z,\overline{\alpha})$, where $s = |z-x|$, $\alpha = (z-x)/|z-x|$, then $\nabla_x \phi \neq 0$ at every point on the surface $\phi = 0$. Thus $\phi(t,x,z)$ satisfies the hypothesis required in [**3**, p. 313] and the following formula holds (for fixed z):

$$\frac{\phi(t,x;z)_+^{-(n-1-j)}}{\Gamma(-n-2-j)} = \frac{(-1)^{n-2-j}}{(n-2-j)!}\delta^{n-2-j}(\phi(t,x,z)).$$

Finally, we have

$$\begin{aligned} I_+^{(n-1)/2}\delta^{(n-2-j)}(\phi) &= (-1)^{n-2-j}\int_0^{\phi} \frac{\tilde{\phi}_+^{-n+1+j}}{\Gamma(-n-2-j)}(\tilde{\phi}-\phi)^{(n-1)/2-1}\,d\tilde{\phi} \\ &= \Gamma\left(\frac{n-1}{2}\right)\phi_+^{-(n-1)/2+j}, \end{aligned}$$

which follows by analytic continuation of the same formula in the obvious case for the one-dimensional analytic family of distributions.

We can do the same in the even case, starting with the formula

$$\delta(x-z) = \frac{(-1)^{n/2}(n-1)!}{(2\pi)^n}\int_{\Omega_{n-1}} \log^{(n)}(\omega(x-z))\,d\omega$$

(see [**9**]).

PROOF OF COROLLARY 6. By Theorem 5,

$$\begin{aligned} u(t,x) = \sum &\left\langle t^{-(n-1)/2}\frac{(\phi(t,x,z))_+^{-(n-1)/2+j}}{\Gamma(-(n-3)/2+j)}\tilde{a}_j(t,x,z), f(z)\right\rangle_z \\ &+ \int_{\mathbf{R}^n} R_N(t,x,z)f(z)\,dz, \end{aligned}$$

where $\langle\ ,\ \rangle_z$ denotes the value of the distribution for the test function $f(z)$.

Finally,

(A.7)

$$\begin{aligned} \left(t - \frac{|x-z|}{r(t,x,\overline{x-z})}\right)_+^{-(n-1)/2+j} &= \left(t - \frac{|x-z|}{r(t,x,\overline{x-z})}\right)_+^{-(n-1)/2+j}\varphi_t(x,x-z) \\ &+ \left(t - \frac{|x-z|}{r(t,x,\overline{x-z})}\right)^{-(n-1)/2+j}(1-\varphi_t(x,x-z)), \end{aligned}$$

where $\overline{x-z} = (x-z)/|x-z|$, φ is a C^{∞} function supported in $|y| < \frac{1}{2}$, $\varphi \equiv 1$ in $|y| < \frac{1}{4}$, and

$$\varphi_t(x,x-z) = \varphi\left(\frac{x-z}{r_t(x)}\right), \quad \text{where } r_t(x) = \min\{r(t,x,\overline{y}) : y \in \Omega_{n-1}\}.$$

The first function in (A.7) is a bounded continuous function whose support is contained in a ball of radius $O(t)$. Hence this can be absorbed by the remainder integral. On the other hand,

$$\begin{aligned} &t^{-(n-1)/2}\left(t-\frac{|x-z|}{r(t,x,\overline{x-z})}\right)_+^{-(n-1)/2+j}(1-\varphi_t(x,x-z)) \\ &\quad = t^{-j}\left(t^2-\frac{|x-z|^2}{r^2(t,x,(x-z)/|x-z|)}\right)^{-(n-1)/2+j} b_j(t,x,z), \end{aligned}$$

where b_j is a C^∞ function for $x,\ z \in \mathbf{R}^n$, and $t \in (0,t_0)$ and bounded uniformly in t for $t \in (0,t_0)$. This proves the corollary.

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